4.1 Let $\mathcal{M}$ be a differentiable manifold and ∇ a connection on $\mathcal{M}$.

(a) Show that there exists no $(1,2)$-type tensor field A on $\mathcal{M}$ with the property that, in any local coordinate system $(x^1,\dots,x^n)$ on $\mathcal{M}$

$$A^k_{ij} = \Gamma^k_{ij}.$$

Hint: Check how Γ^k_{ij} transforms under changes of coordinates.

(b) Show that the torsion $T:\Gamma(\mathcal{M})\times\Gamma(\mathcal{M})\to\Gamma(\mathcal{M})$ of the connection ∇, which is defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

is a tensor field.

(c) Let $\bar{\nabla}$ be a (possibly) different connection on $\mathcal{M}$. Show that the difference $\nabla - \bar{\nabla} : \Gamma(\mathcal{M})\times\Gamma(\mathcal{M})\to\Gamma(\mathcal{M})$ is also a tensor field. Deduce that, there exists a $(1,2)$-type tensor field A such that, in any given local coordinate system $(x^1,\dots,x^n)$,

$$A^k_{ij} = \Gamma^k_{ij} - \bar{\Gamma}^k_{ij}$$

where Γ^k_{ij} and $\bar{\Gamma}^k_{ij}$ are the Christoffel symbols of ∇ and $\bar{\nabla}$, respectively.

(d) Show that, if $h_1, h_2 \in C^\infty(\mathcal{M})$, then $h_1\nabla + h_2\bar{\nabla}$ is again a connection if and only if $h_1 + h_2 = 1$.

4.2 Let $\mathcal{M}$ be a smooth manifold equipped with a connection ∇. We can extend the connection ∇ to a map $\nabla : \Gamma(M)\times Ten^k_l(\mathcal{M})\to Ten^k_l(\mathcal{M})$ (where $Ten^k_l(\mathcal{M})$ is the space of tensor fields on $\mathcal{M}$ of type (k,l)) by the requirements that

- ∇ satisfies the Leibniz rule with respect to tensor products, i.e. for all $X\in\Gamma(M)$

$$\nabla_X(f\otimes g) = \nabla_X f\otimes g + f\otimes\nabla_X g,$$

- ∇ commutes with contractions, i.e.

$$\nabla_X(\mathrm{tr}A) = \mathrm{tr}\big(\nabla_X A\big).$$

Show that, in any local coordinate chart $(x^1,\dots,x^n)$, if Γ^k_{ij} are the Christoffel symbols of ∇ then, for every 1-form ω:

$$(\nabla_{\frac{\partial}{\partial x^i}}\omega)_j = \partial_i\omega_j - \Gamma^k_{ij}\omega_k.$$

Moreover, for any (k,l)-tensor field T:

$$\begin{aligned}(\nabla_{\frac{\partial}{\partial x^a}}T)^{i_1\dots i_k}{}_{j_1\dots j_l} = \partial_a T^{i_1\dots i_k}{}_{j_1\dots j_l} &+ \Gamma^{i_1}_{ab}T^{bi_2\dots i_k}{}_{j_1\dots j_l} + \dots + \Gamma^{i_k}_{ab}T^{i_1\dots i_{k-1}b}{}_{j_1\dots j_l}\\ &- \Gamma^b_{aj_1}T^{i_1i_2\dots i_k}{}_{bj_2\dots j_l} - \dots - \Gamma^b_{aj_l}T^{i_1\dots i_{k-1}i_k}{}_{j_1\dots j_{l-1}b}.\end{aligned}$$

4.3 Let $\mathcal{M}^n$ be a differentiable manifold.

(a) Show that, for any $X, Y, Z \in \Gamma(\mathcal{M})$:

$$\mathcal{L}_{[X,Y]}Z = \mathcal{L}_X\mathcal{L}_Y Z - \mathcal{L}_Y\mathcal{L}_X Z.$$

Show that the above relation also holds when Z is replaced by any tensor field f of type (k,l), $k,l \in \mathbb{N}$. *(Hint: Check how $\mathcal{L}_X$ behaves on tensor products of the form $f_1 \otimes f_2$.)*

(b) Let g be a Riemannian metric on $\mathcal{M}$. We will say that a vector field $X \in \Gamma(\mathcal{M})$ is a *Killing field* if it generates a flow of *isometries* for g, i.e. if, for any $p \in \mathcal{M}$, the flow map $\Phi : (-\delta, \delta) \times \mathcal{U} \to \mathcal{M}$ associated to X in a neighborhood $\mathcal{U}$ of p satisfies

$$(\Phi_t)^*(g \circ \Phi_t) = g \quad \text{for all } t \in (-\delta, \delta).$$

Show that

$$\mathcal{L}_X g = 0.$$

Show also that, in any local system of coordinates, the above equation takes the form

$$g_{ik}\partial_j X^k + g_{jk}\partial_i X^k + \partial_k g_{ij} X^k = 0$$

(*Hint: Apply the product rule on the expression $X(g(Y,Z)) = \mathcal{L}_X(g(Y,Z))$ for suitably chosen vector fields Y, Z.*)

(c) Show that the space $\mathcal{K}$ of Killing fields on $(\mathcal{M}, g)$ is closed under commutation, i.e. that $[X,Y] \in \mathcal{K}$ if $X, Y \in \mathcal{K}$; thus, $\mathcal{K}$ forms a Lie subalgebra of $\Gamma(\mathcal{M})$.

*(d) We will later prove in class that if there exists a point $p \in \mathcal{M}$ and a local system of coordinates around p such that $X|_p = 0$ and $\partial_i X^j|_p = 0$ for all $i, j = 1, \dots, n$, then X vanishes everywhere on the connected component of $\mathcal{M}$ containing p. Using this fact, can you show that on a connected Riemannian manifold $(\mathcal{M}, g)$ the dimension of $\mathcal{K}$ is at most $\frac{n(n+1)}{2}$? Can you find a basis for the Killing algebra $\mathcal{K}$ on $(\mathbb{R}^n, g_E)$?

4.4 Let X, Y be two smooth vector fields on a 2-dimensional manifold $\mathcal{M}$ such that

$$[X, Y] = 0$$

and let $p \in \mathcal{M}$ such that $X|_p, Y|_p$ are *not* collinear. In this exercise, we will show that there exists a local system of coordinates (y^1, y^2) around p so that $X = \frac{\partial}{\partial y^1}$, $Y = \frac{\partial}{\partial y^2}$.

(a) Show that if $\mathcal{U}$ is a neighborhood of p and $\Phi : (-\delta, \delta) \times \mathcal{U} \to \mathcal{M}$ is the flow map associated to X, then, for any $t \in (-\delta, \delta)$ and $q \in \mathcal{U}$:

$$d\Phi_{-t}\big(Y|_{\Phi_t(q)}\big) = Y|_q.$$

(a) Let $\gamma : I \to \mathcal{M}$ be an integral curve of the vector field Y such that $\gamma(0) = p$. Consider the map $\Psi : \Omega \subset \mathbb{R}^2 \to \mathcal{M}$ defined in a neighborhood Ω of 0 defined by the relation

$$\Psi(t, s) = \Phi_t(\gamma(s)).$$

Show that Ψ is a diffeomorphism on its image when restricted to a small neighborhood of 0. Show also that in the coordinate system associated to the chart Ψ^{-1}:

$$X = \frac{\partial}{\partial x^1}, \quad Y = \frac{\partial}{\partial x^2}.$$